Ergodic Theory and Measured Group Theory
Lecture 5

Examples of ergodic transformations.

- Two-sided shift $s: 2^{\mathbb{Z}} \to 2^{\mathbb{Z}}$, $(x_n)_{n\in\mathbb{Z}} \mapsto (x_{n+1})_{n\in\mathbb{Z}}$ with any Bernoulli measure $\nu^{\mathbb{N}}$.
  This is mixing for the same reason as the one-sided shift, hence ergodic.

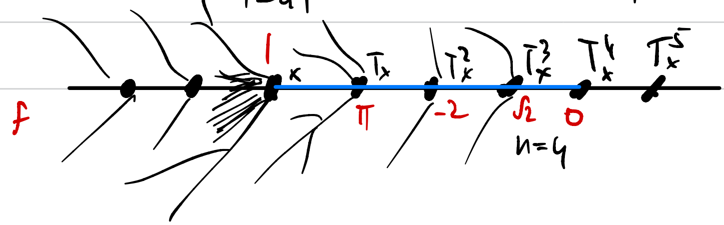
- The Gauss map $g: [0,1) \to [0,1)$, $x \mapsto \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$ with the measure $d\mu = \frac{d\lambda}{\log 2\,(x+1)}$.
  One can show that this map too is mixing, hence ergodic.

Ergodic theorems.

For a transformation $T: (X,\mu) \to (X,\mu)$, where (X,μ) is a st. prob. space, let $I_n := \{0,1,2,\dots,n\} \subseteq \mathbb{N}$, $I_n^T(x) := \{x, Tx, T^2x, \dots, T^n x\}$, and for a function $f: X \to \mathbb{R}$,

$$A_n^T f(x) := \text{the average of } f \text{ over } I_n^T(x), \text{ i.e. } \frac{f(x) + f(Tx) + \dots + f(T^n x)}{|I_n|} = \frac{\sum_{i \leq n} T^i f(x)}{n+1}.$$

We'll omit T from I_n^T and A_n^T if it's clear.



Classical pointwise ergodic theorem (Birkhoff 1931). Let T be a pmp transformation on a st. prob. sp. (X,μ). If T is ergodic then for every $f \in L^1(X,\mu)$,

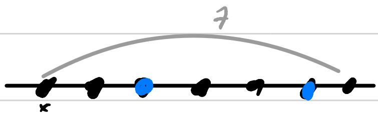
$$\lim_{n\to\infty} A_n^T f(x) = \int f \, d\mu, \quad \text{a.e. } x \in X$$

In fact,

Theorem. For a pmp T, TFAE:

(1) T is ergodic.

(2) $\forall f \in L^1$, $\lim_{n\to\infty} A_n^T f = \int f \, d\mu$ a.e.



(2') $\forall f \in L^\infty$, $\lim_{n\to\infty} A_n^T f = \int f \, d\mu$ a.e.

(2'') $\forall$ meas. set $B \subseteq X$, $\lim_{n\to\infty} \dfrac{|\{x, Tx, \dots, T^n x\} \cap B|}{n+1} = \mu(B)$ for a.e. $x \in X$.

Proof. (1) $\Rightarrow$ (2). The ptwise erg. theorem.

(2) $\Rightarrow$ (2'). Trivial.

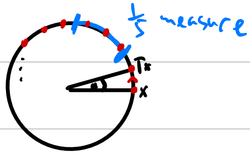
(2') $\Rightarrow$ (2''). Trivial, take $f := \mathbb{1}_B$.

(2'') $\Rightarrow$ (1). Let B be an T-invariant set. Then $A_n \mathbb{1}_B(x) \in \{0,1\}$ for every $x \in X$. Thus, the $\lim_{n\to\infty} A_n \mathbb{1}_B(x) \in \{0,1\}$, hence $\mu(B) \in \{0,1\}$. $\square$

Remark. I don't know how to prove (2'') $\Rightarrow$ (2') or (2') $\Rightarrow$ (2) directly. Thus, some of the power ptwise ergodic theorem lies the ability of dealing with the whole L^1 (unbounded functions).

Applications.

- For irrational rotation $T_\alpha : S^1 \to S^1$, the theorem says that if $I \subseteq S^1$ is a segment of measure $\frac{1}{5}$, then the frequency of a.e. x's trajectory visiting I converges to $\frac{1}{5}$.



- Recall that the baker's map $b_2 : [0,1) \to [0,1)$, $x \mapsto 2x \bmod 1$ with Lebesgue measure is isomorphic to the one-sided shift $s : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ with measure $\{\frac{1}{2}, \frac{1}{2}\}^{\mathbb{N}}$, hence is mixing, hence ergodic.
Likewise, the 10 version of baker's map
$b_{10} : [0,1) \to [0,1)$, $x \mapsto 10x \bmod 1$ is isomorphic to the shift on $10^{\mathbb{N}}$ with measure $\{\frac{1}{10}, \frac{1}{10}, \dots, \frac{1}{10}\}^{\mathbb{N}}$, hence ergodic.

Question. For $x \in [0,1)$, what is frequency of occurance of a given digit, say 7, in the decimal representation of x? I.e., $x = 0.x_0x_1\dots$

$$\lim_{n\to\infty} \frac{\# i : x_i = 7,\ i \le n}{n+1}$$

does this limit exist and what is it?

We apply the ptwise ergodic thm to the function $\mathbb{1}_{\{x_0=7\}}$, i.e. $\mathbb{1}_{[0.7,0.8)}$ for the b_{10} transformation:

$$\frac{\# i : x_i = 7,\ i \leq n}{n+1} = \frac{\mathbb{1}_{[0.7,0.8)}(x) + \mathbb{1}_{[\,)}(b_{10}x) + \dots + \mathbb{1}_{[\,)}(b_{10}^n x)}{n+1}$$

$$\xrightarrow[n\to\infty]{} \lambda\big([0.7, 0.8)\big) = \frac{1}{10} \quad \text{a.e. } x \in [0,1).$$

- Question. What is the frequency of the occurance of a given natural number $k \in \mathbb{N}$ in the continued fraction expansion of a real in $[0,1)$?
  Let $x := [x_0, x_1, x_2, \dots]$ be the cont. frac. exp. of $x \in [0,1)$; does the limit

  $$\lim_{n\to\infty} \frac{\# i \leq n : x_i = k}{n+1}$$ exist and what is it?

Recall that the Gauss map $g : [0,1) \to [0,1)$ maps $[x_0, x_1, x_2 \dots]$ to $[x_1\ x_2\ x_3 \dots]$. Let $f := \mathbb{1}_{\{x_0 = k\}} = \mathbb{1}_{(\frac{1}{k+1}, \frac{1}{k}]}$ and apply ergodic theorem to the Gauss map g and $\mathbb{1}_{(\frac{1}{k+1}, \frac{1}{k}]}$:

$$\lim_{n\to\infty} \frac{\# i \le n : x_i = k}{n+1} = \lim_{n\to\infty} \frac{\left|\{x, gx, g^2x, \dots, g^n x\} \cap [\frac{1}{k+1}, \frac{1}{k})\right|}{n+1} \overset{a.e.\,x}{=} \mu\left(\left(\tfrac{1}{k+1}, \tfrac{1}{k}\right]\right)$$

$$= \int_{1/(k+1)}^{1/k} \frac{1}{\log 2\,(x+1)}\, d\lambda(x) = \frac{1}{\log 2}\left(\log\left(\tfrac{1}{k}\right) - \log\left(\tfrac{1}{k+1}\right)\right) \sim \frac{1}{\log 2} \log\frac{(k+1)^2}{k(k+2)}.$$

$(1 \sim 41.56\%, 2 \sim 16.99\%, 3 \sim 9.31\%, 4 \sim 5.89\%, \ \textit{etc.})$

Further applications to Gauss map:

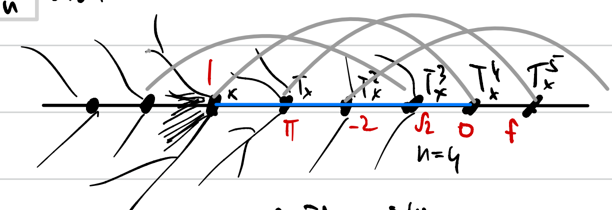
- $\lim_{n\to\infty} \left(\prod_{i=0}^{n} x_i\right)^{\frac{1}{n+1}} = \prod_{k=0}^{\infty} \left(\frac{(k+1)^2}{k(k+2)}\right)^{\log k/\log 2}$. Take $f(x) := \sum_{k=0}^{\infty} (\log k)\, \mathbb{1}_{(\frac{1}{k+1}, \frac{1}{k}]}$.
- $\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} x_i = \infty$. $\frac{1}{n+1}\sum_{i=0}^{n} x_i \ge \frac{1}{N+1}\sum_{n<N} g^n f_M(x)$, where $f_M(x) = \sum_{k\le M} k\, \mathbb{1}_{(\frac{1}{k+1}, \frac{1}{k}]}$.

Proof of the ptwise ergodic theorem (Keane-Petersen, Ts.).

Let T be a pmp ergodic transformation on (X, μ).

- **Bridge lemma.** For any $f \in L^1(X, \mu)$, $\forall n \in \mathbb{N}$,

$$\text{global} \longrightarrow \boxed{\int f\, d\mu} = \int \boxed{A_n^T f}\, d\mu \longleftarrow \text{local}$$



Proof. By pmp, the change-of-var holds: $\int f\, d\mu = \int Tf\, d\mu = \dots = \int T_n f\, d\mu$

Thus $\int (f + Tf + \dots + T_n f)\, d\mu = (n+1)\int f\, d\mu$, so $\int f\, d\mu = \int \frac{f + Tf + \dots + T^n f}{n+1}\, d\mu = \int A_n f\, d\mu$. □

- By subtracting the constant $\int f \, d\mu$ from f, we may assume that $\int f \, d\mu = 0$. Thus, to prove $\lim_n A_n^T f = 0$ a.e. it's enough to show that
$\limsup_n A_n^T f \leq 0$ and $\liminf_{n \to \infty} A_n^T f \geq 0$. We'll only do the first because of symmetry.

- Suppose towards a contradiction that $\limsup A_n f > 0$ on a positively-measured set. For such x,